1. Consider $n!$, since half of the factors of $n!$ is at least $\frac{n}{2}$

$$n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$$

$$\Rightarrow \sqrt[n]{n!} \geq \left(\frac{n}{2}\right)^{\frac{1}{2}}$$

Then $\lim \sqrt[n]{n!} = +\infty$ ← add the detail yourself

2.

$$\left|\frac{z_1+z_2+z_3+\dots+z_n}{n} - A\right| = \left|\frac{(z_1-A)+(z_2-A)+\dots+(z_n-A)}{n}\right|$$

$$\leq \left|\frac{z_1-A}{n}\right| + \dots + \left|\frac{z_n-A}{n}\right|$$

(more detail of ε-N notation)

$$\lim_n \frac{z_1+z_2+\dots+z_n}{n} = A$$

3 a) Note b_n is decreasing and bounded

c_n is increasing and bounded

b) $c_n \leq b_n$

$\Rightarrow \lim c_n \leq \lim b_n$

$\Rightarrow \underline{\lim}\, a_n \leq \overline{\lim}\, a_n$

4. similar to 3

5. Book P. 83 Thm 3.4.11 (b)

6. Thm 3.4.11 (d)

2. Let $\varepsilon > 0$. Since $\lim z_n = A$, (z_n) is bounded

Let $M = \sup\{|z_n|\}$.

$\exists\, N_0 \in \mathbb{N}$ s.t. $|z_n - A| < \frac{\varepsilon}{2}$, if $n \geq N_0$

Let $N = \max\left\{ N_0, \frac{2(N_0-1)(M+|A|)}{\varepsilon} \right\}$

Then if $n \geq N$, we have

$$\left| \frac{z_1 + \dots + z_n}{n} - A \right| = \left| \frac{(z_1 - A) + (z_2 - A) + \dots + (z_n - A)}{n} \right|$$

$$\leq \underbrace{\left| \frac{(z_1 - A) + \dots + (z_{N_0-1} - A)}{n} \right|}_{R_1} + \underbrace{\left| \frac{(z_{N_0} - A) + \dots + (z_n - A)}{n} \right|}_{R_2}$$

$$R_1 < \left| \frac{\overbrace{(M+|A|) + (M+|A|) + \dots + (M+|A|)}^{N_0 - 1\text{'s}}}{n} \right| = \frac{(N_0-1)(M+|A|)}{n}$$

Since $n \geq N \geq \frac{2(N_0-1)(M+|A|)}{\varepsilon}$,

$$R_1 < \frac{\varepsilon}{2}$$

$$R_2 < \frac{(n-N_0+1)\frac{\varepsilon}{2}}{n} \leq \frac{\varepsilon}{2} \quad \text{since } |z_m - A| < \frac{\varepsilon}{2} \text{ if } m \geq N_0$$

Hence $\left| \frac{z_1 + \dots + z_n}{n} - A \right| \leq R_1 + R_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ if $n > N$

Then $\lim\limits_n \frac{z_1 + \dots + z_n}{n} = A$

7. "(a_n) converge $\Leftarrow$ $\overline{\lim}\, a_n = \underline{\lim}\, a_n$"

Let $\overline{\lim}\, a_n = \alpha$, $\underline{\lim}\, a_n = \alpha = \beta$

by 5a(i) $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$,

$$a_n < \alpha + \varepsilon$$

5b (i) $\forall \varepsilon > 0$ $\exists N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$,

$$\alpha - \varepsilon < a_n$$

Then let $\varepsilon > 0$, take $N = \max\{N_1, N_2\}$

if $n \geq N$, $-\varepsilon < a_n - \alpha < \varepsilon$

$$\Rightarrow \quad |a_n - \alpha| < \varepsilon$$

$\Rightarrow \quad \lim a_n = \alpha$

"(a_n) converges $\Rightarrow$ $\overline{\lim}\, a_n = \underline{\lim}\, a_n$"

By 6, $\because \underset{\overline{\lim}\, a_n}{\overset{\parallel}{\alpha}} \in E$, $\exists (a_{n_k})$ subsequence of (a_n) s.t. $a_{n_k} \to \alpha$

Similarly, for $\underline{\lim}\, a_n = \beta$, $\exists (a_{m_k})$ subsequence of (a_n) s.t. $a_{m_k} \to \beta$

But, any subsequence of (a_n) converges to $\lim a_n$

since (a_n) converge

Then $\underline{\lim}\, a_n = \alpha = \lim a_n = \beta = \overline{\lim}\, a_n$

8. Let $\varepsilon > 0$, Since (s_n) is Cauchy sequence,

$\exists N \in \mathbb{N}$ s.t. if $m, n > N$, $|s_m - s_n| < \varepsilon$,

Choose this N,

Then $\big||s_m| - |s_n|\big| \le |s_m - s_n| < \varepsilon$, if $m, n > N$

Then $(|s_n|)$ is Cauchy sequence, thus converges.

The converse is not true,

for example, $s_n = (-1)^n$

9. By MI, $\sqrt{2} < s_n < 2 \quad \forall n$ — ①

(Since $\sqrt{2} < 2$, $s_{n+1} < \sqrt{2+2} = 2$)

$s_{n+1}^2 - 2 = s_n$

$\Rightarrow s_{n+1}^2 - s_{n+1} - 2 = s_n - s_{n+1}$

Since $(s_{n+1} - 2)(s_{n+1} + 1) < 0$ by ①

$\Rightarrow s_{n+1} > s_n$

Then (s_n) is increasing bounded sequence, so converges

and let $\lim s_n = s$, $s^2 = 2 + s \Rightarrow s = 2$

10. Since $\overline{\lim}\, a_n \neq +\infty$, we can assume (a_n) is bounded abo[ve]

By 6, let $(a_{n_k} + b_{n_k})$ be subsequence of $(a_n + b_n)$ s.t.

$$\lim_k (a_{n_k} + b_{n_k}) = \overline{\lim_n}\, (a_n + b_n)$$

By 6 again let $(a_{n_{k_m}})$ be subsequence of (a_{n_k}) s.t.

$$\lim_m a_{n_{k_m}} = \overline{\lim_k}\, a_{n_k}$$

10. Since $(a_{n_k}+b_{n_k})$ converge,

$(a_{n_{k_m}}+b_{n_{k_m}})$ subsequence of $(a_{n_k}+b_{n_k})$ also converge

that is $\lim_m (a_{n_{k_m}}+b_{n_{k_m}}) = \lim_k (a_{n_k}+b_{n_k}) = \overline{\lim}(a_n+b_n)$

Since (a_n) is bounded above, (a_{n_k}) is bounded above

$\Rightarrow \limsup_k a_{n_k}$ is finite

exist and not infinite

Also, $\lim_m b_{n_{k_m}} = \lim_m (a_{n_{k_m}}+b_{n_{k_m}}) - \lim_m a_{n_{k_m}}$

Then let $\alpha = \lim_m a_{n_{k_m}}$ ($\because$ $(a_{n_{k_m}})$, $(b_{n_{k_m}})$ converge)

$\beta = \lim_m b_{n_{k_m}}$

$\Rightarrow \alpha+\beta = \lim_m (a_{n_{k_m}}+b_{n_{k_m}}) = \overline{\lim}(a_n+b_n)$

Since $(a_{n_{k_m}})$ is subsequence of (a_n)

$(b_{n_{k_m}})$ is " (b_n)

by 6, $\alpha \le \overline{\lim}\, a_n$, $\beta \le \overline{\lim}\, b_n$

$\overline{\lim}(a_n+b_n) = \alpha+\beta \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$

11. similar to Q9.